GSPT beyond the standard form

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Overview

- Geometric Singular Perturbation Theory
- Glycolythic Relaxation Oscillations
- Mitotic Oscillator
- Conclusion and Outlook

joint work with Ilona Kosiuk (MPI Leipzig)

Singularly perturbed systems in standard form

$$\begin{aligned} x' &= \varepsilon f(x,y,\varepsilon) \\ y' &= g(x,y,\varepsilon) \end{aligned}$$

x slow, y fast, $\varepsilon << 1$,
fast time scale τ, $\quad ' = \frac{d}{d\tau}$
transform to slow time scale $t := \varepsilon\tau$

$$\begin{aligned} \dot{x} &= f(x,y,\varepsilon) \\ \varepsilon\dot{y} &= g(x,y,\varepsilon) \end{aligned}$$

- global separation into slow and fast variables
- singular behaviour with respect to one parameter

Limiting systems for $\varepsilon = 0$

- layer problem
$$\begin{aligned} x' &= 0 \\ y' &= g(x, y, 0) \end{aligned}$$

- reduced problem
$$\begin{aligned} \dot{x} &= f(x, y, 0) \\ 0 &= g(x, y, 0) \end{aligned}$$

- critical manifold $S := \{g(x, y, 0) = 0\}$

S is a manifold of equilibria for layer problem.

reduced problem is a dynamical system on S.

Geometric Singular Perturbation Theory (GSPT)

critical manifold S normally hyperbolic, i.e. $\frac{\partial g}{\partial y}|_S$ hyperbolic $\Rightarrow$ S perturbs smoothly to slow manifold S_ε for ε small, $\exists$ stable- and unstable manifolds $W^s(S_\varepsilon)$, $W^u(S_\varepsilon)$, invariant foliations,.... (Fenichel, 1979)

Refinements: Exchange Lemma, Fenichel Normal Form,... (C. Jones, N. Kopell, T. Kaper, P. Brunovsky,...)

Applications: analysis of periodic, heteroclinic, homoclinic, and chaotic dynamics; existence and stability of travelling waves,...

Extensions: blow-up method at non-hyperbolic parts of S

Phenomena:

- relaxation oscillations
- canard solutions
- mixed-mode oscillations
- delayed bifurcations

F. Dumortier, R. Roussarie,

M. Krupa, M. Wechselberger, P. Sz.,...

More difficulties

in many applications, e.g. from biology and chemistry, there is

- no global separation into slow and fast variables
- dynamics on more than two distinct time-scales
- singular or non-uniform dependence on several parameters
- several scaling regimes with different limiting problems are needed

Claim: GSPT and in particular the blow-up method is useful for such problems.

Strategy

- identify fastest time-scale and corresponding scale of dependent variables, rescale
- often the limiting problem has a (partially) non-hyperbolic critical manifold
- use (repeated) blow-ups to desingularize
- identify relevant singular cycles, etc.
- carry out perturbation analysis

case studies:

Autocatalator, Glykolythic Oscillations,

Mitotic Oscillator   with I. Kosiuk (MPI Leipzig)

Glycolytic Oscillations

- model for glycolythic oscillations
- relaxation oscillations
- two parameter singular perturbation problem: $\varepsilon, \delta << 1$
- various scaling regimes
- $(\varepsilon, \delta) = (0, 0)$ very degenerate
- geometric analysis based on blow-up method

Glycolytic Oscillations Model (GOM)

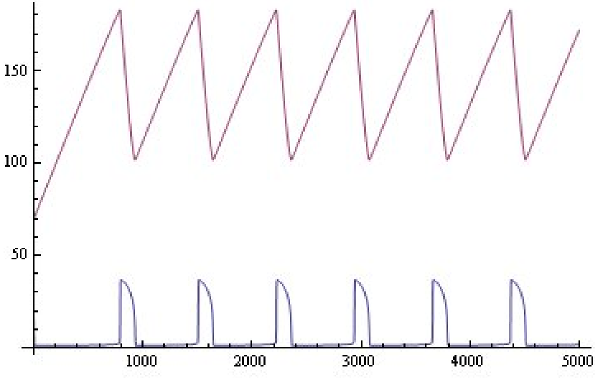
$$\dot{\alpha} = \mu\rho^{-1} - \rho^{-1}\phi(\alpha,\gamma)$$
$$\dot{\gamma} = \lambda\phi(\alpha,\gamma) - \gamma$$

$$\phi(\alpha,\gamma) = \frac{\alpha^2(\gamma+1)^2}{L+\alpha^2(\gamma+1)^2}$$

- glycolysis: complicated biochemical reaction glucose $\rightarrow$ pyruvate
- simplified model
- substrate α, product γ
- parameters: μ, ρ, λ, L
- λ and L large

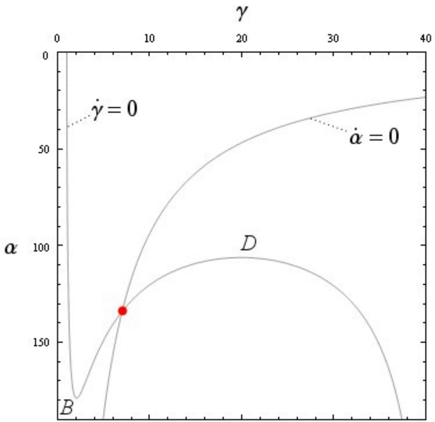
L. Segel, A. Goldbeter, *Scaling in biochemical kinetics: dissection of a relaxation oscillator* J. Math. Biol. (1994)

$L = 5 \times 10^6, \quad \rho = 2.5, \quad \lambda = 40, \quad \mu = 0.15$



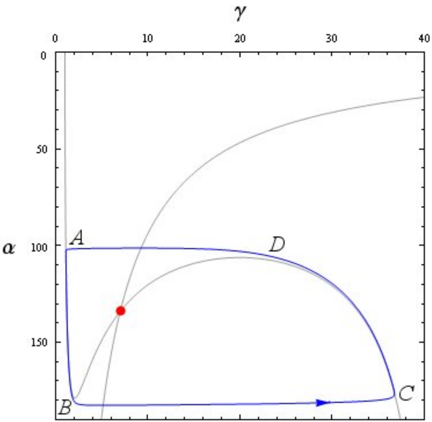
- relaxation oscillation: α, γ

$L = 5 \times 10^6, \quad \rho = 2.5, \quad \lambda = 40, \quad \mu = 0.15$



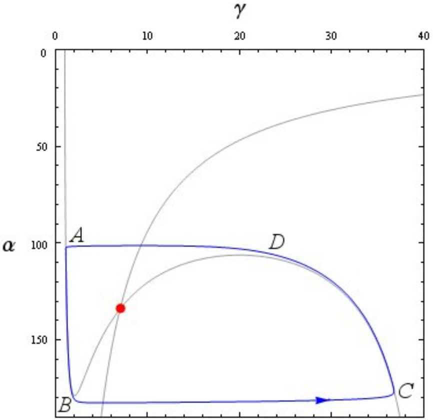
- L large, λ fixed: classical relaxation oscillations

$$L = 5 \times 10^6, \quad \rho = 2.5, \quad \lambda = 40, \quad \mu = 0.15$$



- L large, λ fixed: classical relaxation oscillations

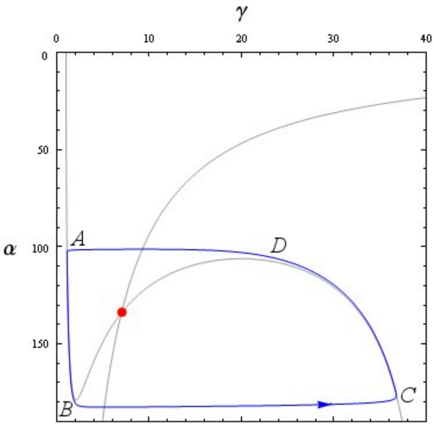
$$L = 5 \times 10^6, \quad \rho = 2.5, \quad \lambda = 40, \quad \mu = 0.15$$



- L large, λ fixed: classical relaxation oscillations
- L, λ both large: more complicated
- small parameter $\varepsilon := \sqrt{\frac{\lambda}{L}}$

Scaling analysis

$$A \approx \left(\frac{2\sqrt{L}}{\lambda}, 1\right), B \approx \left(\frac{1}{2}\sqrt{\frac{L}{\lambda}}, 1\right), C \approx \left(\frac{1}{2}\sqrt{\frac{L}{\lambda}}, \lambda\right), D \approx \left(\frac{2\sqrt{L}}{\lambda}, \frac{\lambda}{2}\right)$$



Goldbeter & Segel:

$$\sqrt{\frac{\lambda}{L}} \ll \frac{1}{\sqrt{\lambda}} \ll 1 \qquad \Longrightarrow$$

relaxation cycle exists

rescaled variables:

$$\alpha = \sqrt{\frac{L}{\lambda}} a, \quad \gamma = \lambda b - 1$$

Rescaled Equations

System (GOM) in (a, b) variables with $\rho = 1$

$$a' = \varepsilon[\mu - \frac{a^2b^2}{\delta^2+a^2b^2}]$$
$$b' = \frac{a^2b^2}{\delta^2+a^2b^2} - b + \delta^2$$

$$\varepsilon := \sqrt{\frac{\lambda}{L}}, \quad \delta := \frac{1}{\sqrt{\lambda}}$$

multiplication with $\delta^2 + a^2b^2 > 0$ gives

$$\begin{aligned} a' &= \varepsilon[a^2b^2(\mu - 1) + \mu\delta^2] \\ b' &= a^2b^2(1 - b) + \delta^2(a^2b^2 - b + \delta^2) \end{aligned} \qquad (1)$$

- slow-fast system in standard form: a slow, b fast
- L large $\Rightarrow$ ε small for λ fixed; λ large $\Rightarrow$ δ small
- Goldbeter-Segel condition: $\varepsilon \ll \delta \ll 1$

Slow-fast subsystems for $\varepsilon = 0$

- **layer problem**

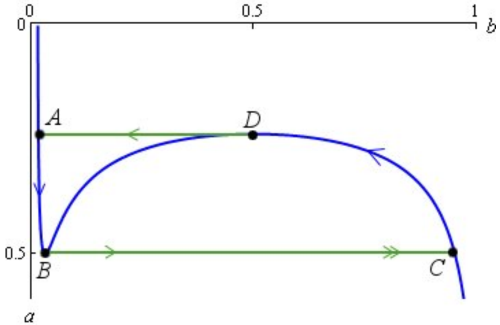
$$\begin{aligned} a' &= 0 \\ b' &= a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) \end{aligned} \tag{2}$$

- **reduced problem**

$$\begin{aligned} \dot{a} &= a^2b^2(\mu - 1) + \mu\delta^2 \\ 0 &= a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) \end{aligned} \tag{3}$$

Critical manifold

$$S^\delta = \{(a,b): \quad a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}$$

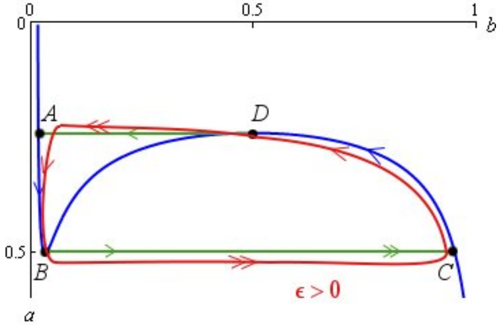


$S^\delta = S_l \cup B \cup S_m \cup D \cup S_r, \quad \delta > 0$

singular cycle Γ_0^δ: four segments AB, BC, CD, DA

Critical manifold

$$S^\delta = \{(a,b) : \quad a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}$$



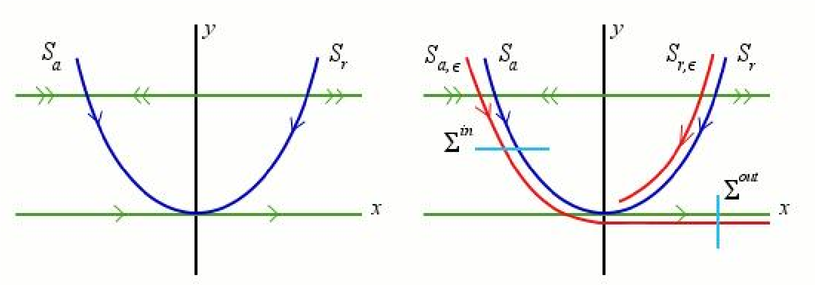
$$S^\delta = S_l \cup B \cup S_m \cup D \cup S_r, \quad \delta > 0$$

$\Rightarrow$ relaxation oscillations for $\delta > 0$ fixed and ε small.

Fold Point: $(0,0)$ nonhyperbolic, blow-up method

Krupa, Sz. (2001)

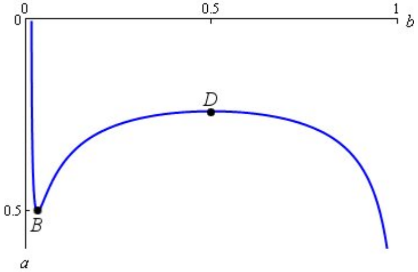
$$\begin{aligned} x' &= -y + x^2 + \cdots \\ y' &= -\varepsilon + \cdots \end{aligned}$$



- asymptotics of $S_{a,\varepsilon} \cap \Sigma^{out}$
- map: $\pi : \Sigma^{in} \to \Sigma^{out}$ contraction, rate $e^{-C/\varepsilon}$

Critical manifold S^δ depends singularly on δ

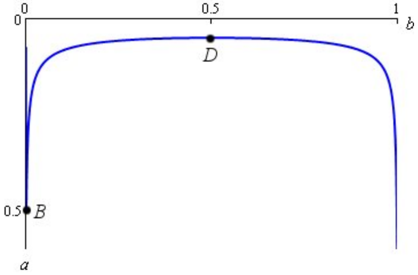
$$S^\delta = \{(a,b): \quad a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}$$



$$\delta = 1/8$$

Critical manifold S^δ depends singularly on δ

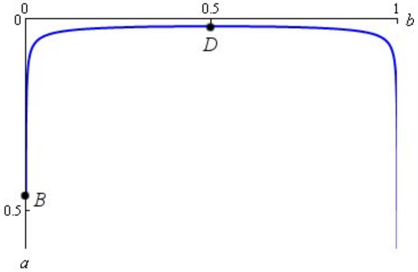
$$S^\delta = \{(a,b): \quad a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}$$



$$\delta = 1/40$$

Critical manifold S^δ depends singularly on δ

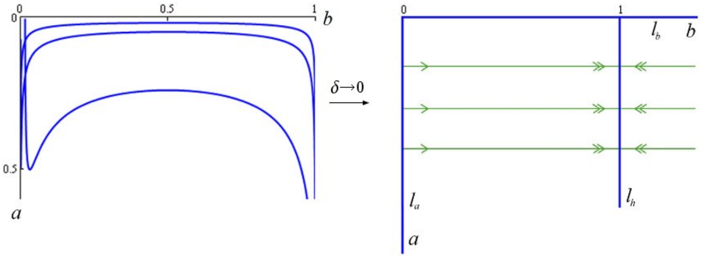
$$S^\delta = \{(a,b): \quad a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}$$



$$\delta = 1/100$$

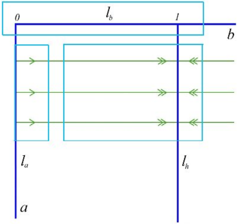
$\delta = 0$, Critical manifold S^0

$$a^2b^2(1-b) = 0, \qquad a = 0, \quad b = 0, \quad b = 1$$



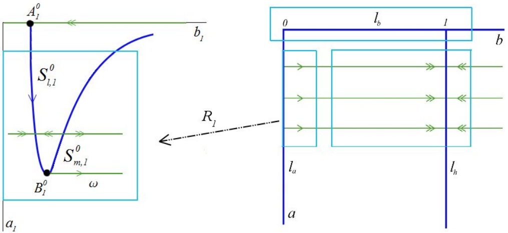
The folded critical manifold S^δ collapses to the more singular "manifold" $S^0 = l_a \cup l_b \cup l_h$

Scaling Regimes



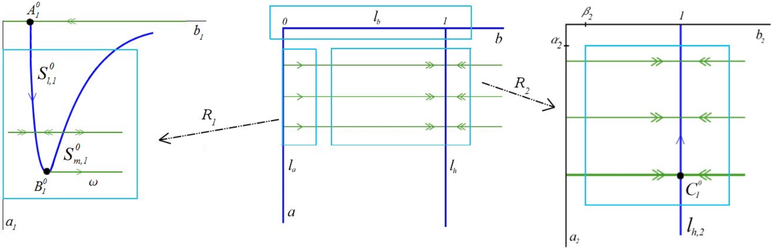
Scaling Regimes:

Regime 1: $a = O(1), b = O(\delta^2)$



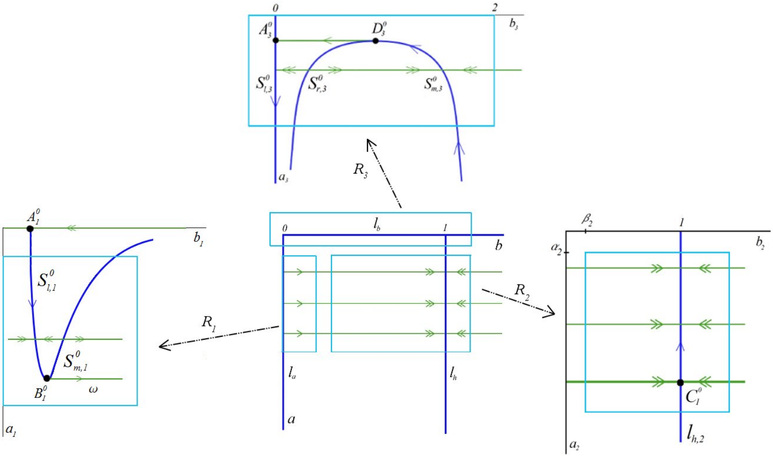
Scaling Regimes:

Regime 1: $a = O(1), b = O(\delta^2)$, Regime 2: $a = O(1), b = O(1)$



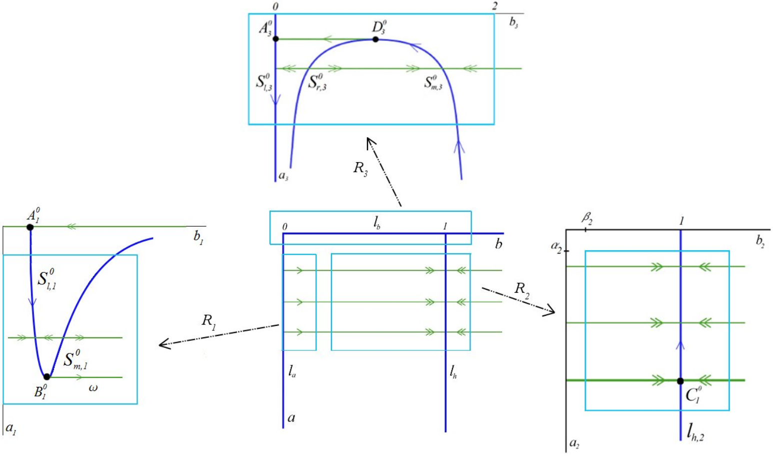
Scaling Regimes: Regime 3: $a = O(\delta), b = O(1)$

Regime 1: $a = O(1), b = O(\delta^2)$, Regime 2: $a = O(1), b = O(1)$



Scaling Regimes: Regime 3: $a = O(\delta), b = O(1)$

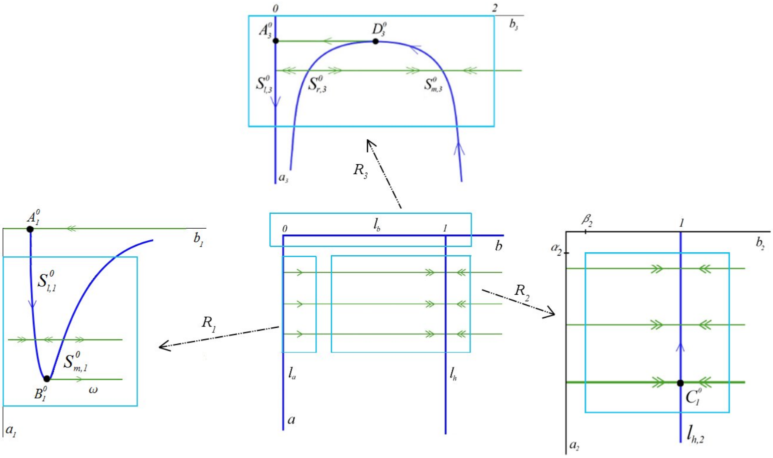
Regime 1: $a = O(1), b = O(\delta^2)$, Regime 2: $a = O(1), b = O(1)$



in Regimes 1 - 3 results for $0 < \varepsilon \ll \delta \ll 1$

Scaling Regimes: Regime 3: $a = O(\delta), b = O(1)$

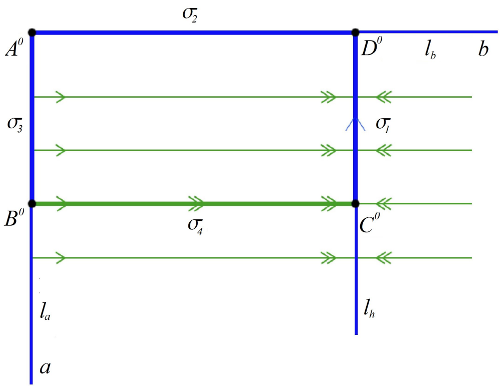
Regime 1: $a = O(1), b = O(\delta^2)$, Regime 2: $a = O(1), b = O(1)$



matching? overlap?

For $\varepsilon = 0$, $\delta = 0$ exists very degenerate singular cycle

singular cycle $\Gamma_0^0 := \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$



lines $a = 0$, $b = 0$ non-hyperbolic

line $b = 1$ hyperbolic

Main result

Theorem:
For $\mu < 1$ there exist $\delta_0 > 0$ and $\tilde{\varepsilon}_0 > 0$ such that system (1) has a unique attracting periodic orbit $\Gamma_\varepsilon^\delta$ for $0 < \delta \leq \delta_0$ and $0 < \varepsilon \leq \tilde{\varepsilon}_0 \delta$ with the properties

1. $\Gamma_\varepsilon^\delta$ tends to singular cycle Γ_0^δ as $\varepsilon \to 0$ for $\delta \in (0, \delta_0]$,
2. $\Gamma_\varepsilon^\delta$ tends to singular cycle Γ_0^0 as $(\delta, \varepsilon) \to (0, 0)$.

Proof: based on repeated blow-ups

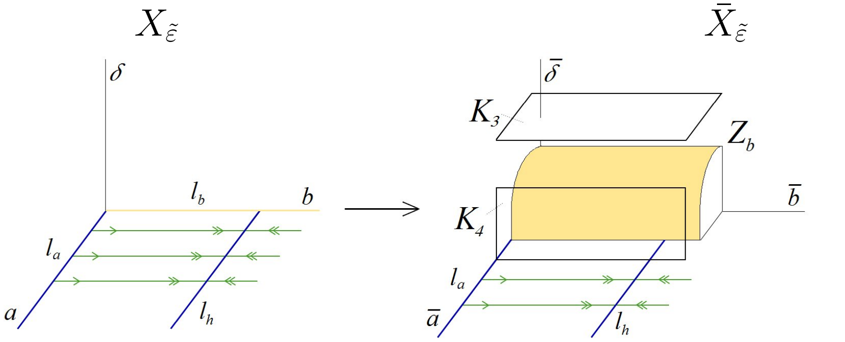
Extended system

because of $\varepsilon \ll \delta \ll 1$ set $\varepsilon := \tilde{\varepsilon}\delta$

$$\begin{aligned} a' &= \tilde{\varepsilon}\delta[a^2b^2(\mu - 1) + \mu\delta^2] \\ b' &= a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) \\ \delta' &= 0 \end{aligned}$$

- three-dimensional vector field $X_{\tilde{\varepsilon}}$ defined on $\mathbb{R}^3$
- $\tilde{\varepsilon}$ is the singular perturbation parameter causing the slow-fast structure
- family of 1-dim. critical manifolds S^δ corresponds to 2-dim. critical manifold S

Cylindrical blow-up of the non-hyperbolic line l_b



$$a = r\bar{a}, \quad b = \bar{b}, \quad \delta = r\bar{\delta}, \qquad (\bar{a}, \bar{\delta}, r, \bar{b}) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$$

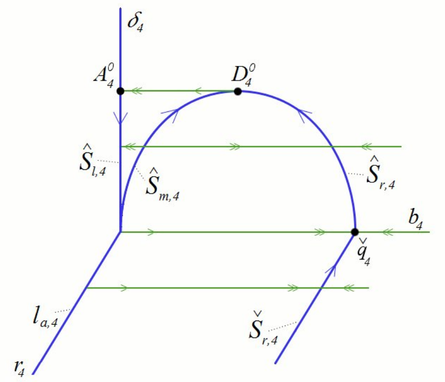
- chart K_3: $\bar{\delta} = 1$, corresponds to Regime 3
- chart K_4: $\bar{a} = 1$, covers Regime 3 and Regime 2

Dynamics in K_4 slow-fast system, $0 < \tilde{\varepsilon} \ll 1$

$$
\begin{aligned}
r_4' &= \tilde{\varepsilon} g_1(b_4, \delta_4, r_4) \\
\delta_4' &= \tilde{\varepsilon} g_2(b_4, \delta_4, r_4) \\
b_4' &= f(b_4, \delta_4, r_4)
\end{aligned}
$$

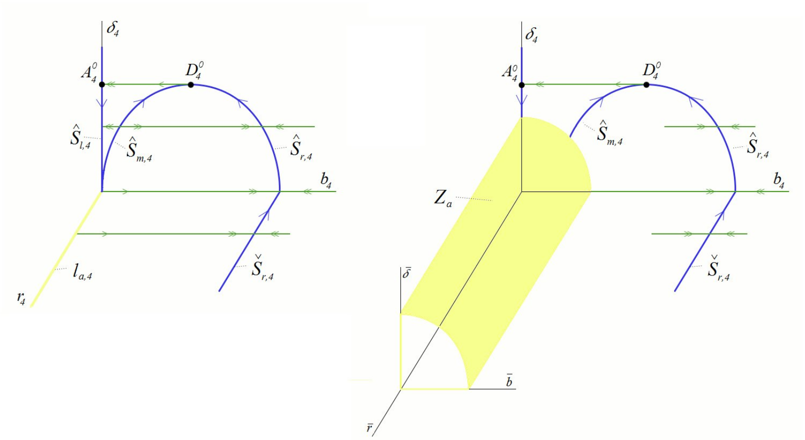
slow variables r_4, δ_4

fast variable b_4



- invariant planes: $r_4 = 0$ and $\delta_4 = 0$
- critical manifold S partially desingularized
- S cusp-like along non-hyperbolic line l_a

Cylindrical blow-up of the non-hyperbolic line $l_{a,4}$



$$r_4 = \bar{r},\ b_4 = \rho^2\bar{b},\ \delta_4 = \rho\bar{\delta}, \quad (\bar{b}, \bar{\delta}, \rho, \bar{r}) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$$

Charts K_1 and K_2

chart K_1: $\bar{\delta} = 1$

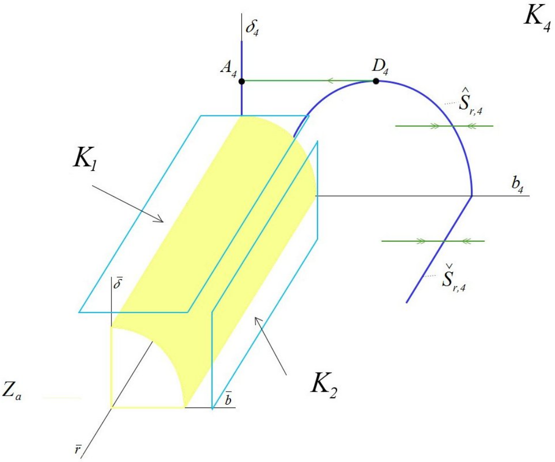
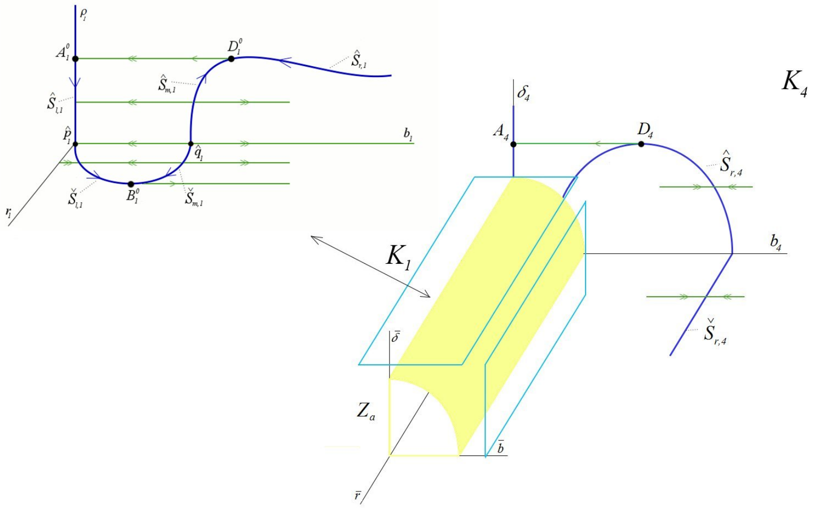


chart K_2: $\bar{b} = 1$

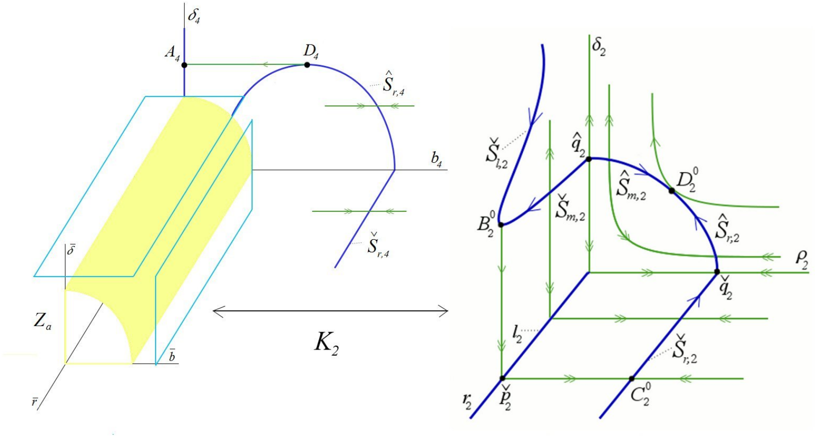
Chart K_1 covers Regime 1 and parts of Regime 3



slow-fast for $\tilde{\varepsilon} \ll 1$, critical manifold S desingularized

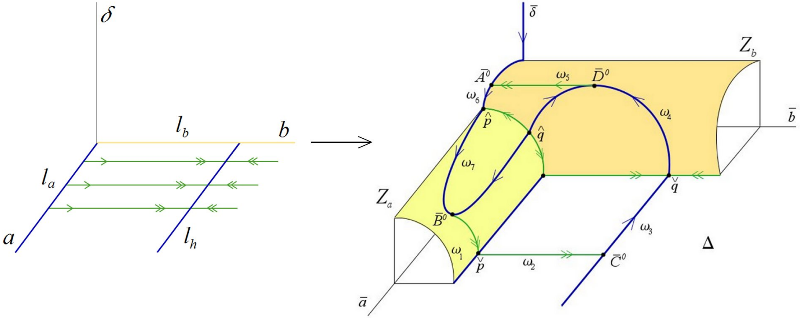
Chart K_2

covers Regime 1, Regime 2, and parts of Regime 3

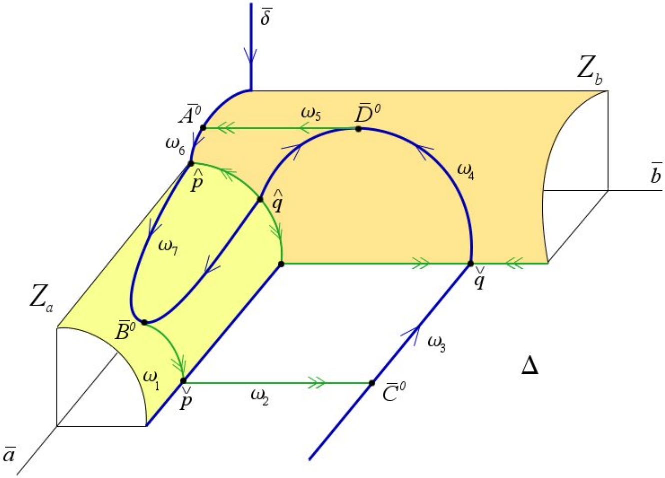


slow-fast for $\tilde{\varepsilon} \ll 1$, critical manifold S desingularized

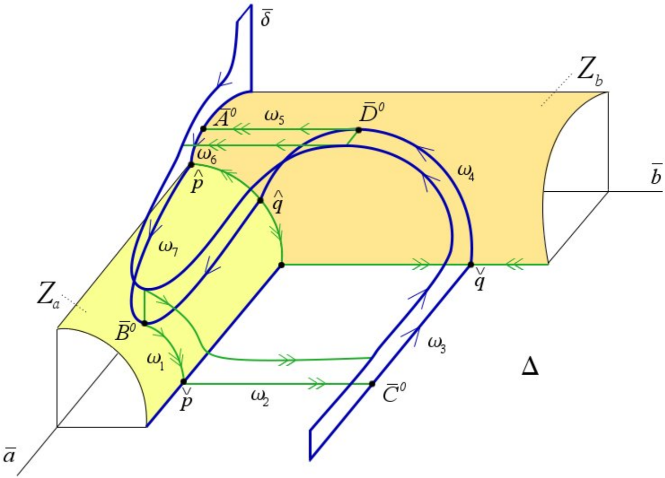
After two cylindrical blow-ups...



Blown-up singular cycle Γ_0^0

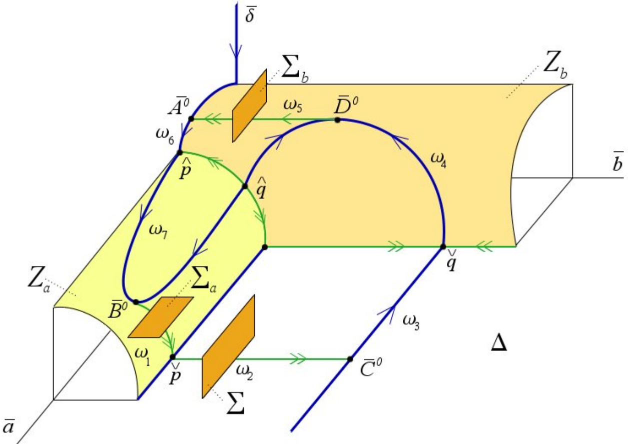


Blown-up critical manifold S and family of singular cycles Γ_0^δ



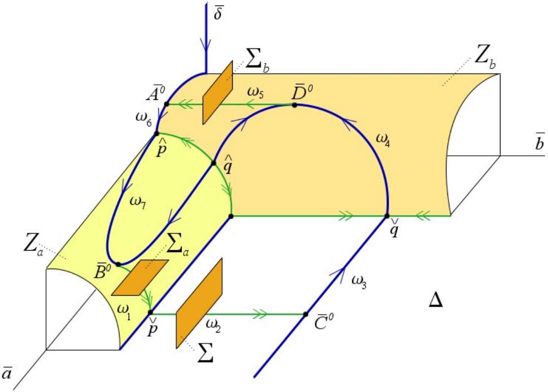
Proof of main result

- Poincaré map $\Pi : \Sigma \to \Sigma$ for δ, $\tilde{\varepsilon}$ small
- S perturbs to slow manifold S_ε for $\tilde{\varepsilon}$ small
- folds, transition near hyperbolic line



Define sections Σ, Σ_b, Σ_a transversal to ω_2, ω_5, ω_1

- maps $\Pi_1 : \Sigma \to \Sigma_b$, $\Pi_2 : \Sigma_b \to \Sigma_a$, $\Pi_3 : \Sigma_a \to \Sigma$
- $\Pi : \Sigma \to \Sigma$, $\Pi := \Pi_3 \circ \Pi_2 \circ \Pi_1$
- attraction to slow manifolds, passage near folds, transition near hyperbolic line

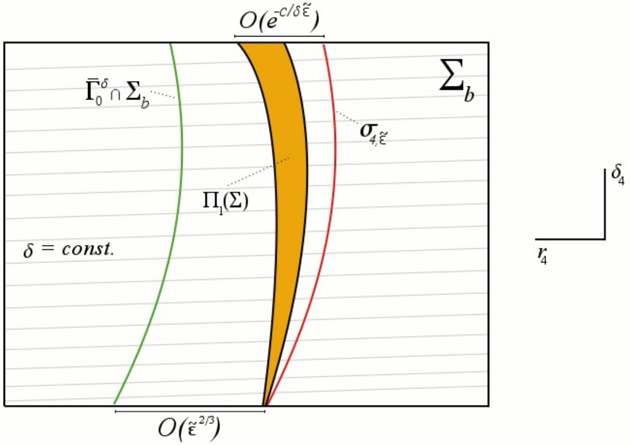


Folds are treated by available results, which are proved by other blow-ups

- all maps are analyzed in the appropriate charts: Π_1 in K_4, Π_2 in K_4, and K_1, Π_3 in K_2
- Π_1 and Π_2 are very similar: exp. strong contractions
- Π_3 desribes passage near a line of hyperbolic equilibria: at most algebraically expanding

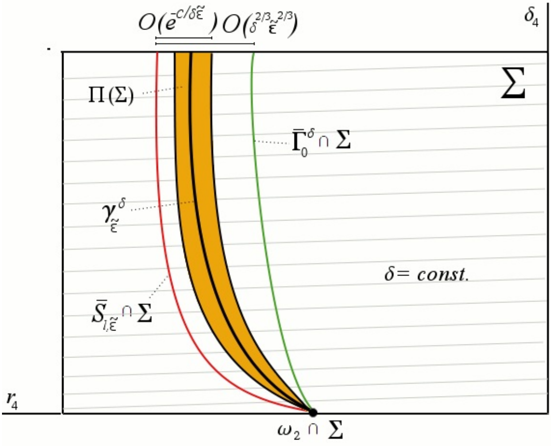
Π_1 maps Σ to an exponentially thin wedge $\Pi_1(\Sigma)$ exp. close to $S_\varepsilon \cap \Sigma_b$

- Π_1 restricted to a leaf $\delta = const.$ is a contracting with rate $e^{-c/\delta\tilde{\varepsilon}}$.



Π maps Σ to an exponentially thin wedge $\Pi(\Sigma)$ exp. close to $S_\varepsilon \cap \Sigma$

- Π restricted to $\delta = const.$ contracts, rate $e^{-c/\delta\tilde{\varepsilon}}$.
- $\Rightarrow \exists$ fixed point of Π, main result is proved

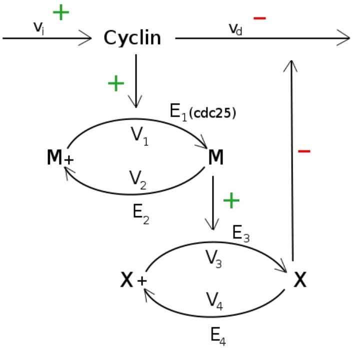


Summary

- identify fastest time-scale and corresponding scale of dependent variables, rescale
- often the limiting problem has a (partially) non-hyperbolic critical manifold
- use (repeated) blow-ups to desingularize
- identify relevant singular cycles, etc.
- carry out perturbation analysis

Mitotic Oscillator

enzyme reaction relevant for cell division cycle



- **Cyclin** triggers the transformation of inactive $(M+)$ into active (M) cdc2 kinase by enhancing the rate of a phosphatase. A kinase with rate v2 reverts this modification.
- **Cdc2 kinase** - phosphorylates a protease shifting it from the inactive $(X+)$ to the active (X) form. The cyclin protease is inactivated by a further phosphatase.

A. Goldbeter, *A minimal cascade model for the mitotic oscillator involving cyclin and cdc2 kinase*. Proc Natl Acad Sci USA 88 (1991), 9107-9111.

Mitotic Oscillator (MO)

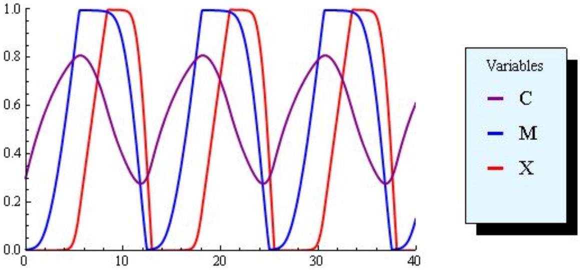
$$
\begin{aligned}
\frac{dC}{dt} &= v_i - v_d X \frac{C}{K_d + C} - k_c C \\
\frac{dM}{dt} &= V_1 \frac{1-M}{K_1 + 1 - M} - V_2 \frac{M}{K_2 + M} \\
\frac{dX}{dt} &= V_3 \frac{1-X}{K_3 + 1 - X} - V_4 \frac{X}{K_4 + X}
\end{aligned}
$$

Michaelis-Menten kinetics

C cyclin concentration, M, X fractions of active kinase and cyclin protease, $1-X$, $1-M$ fractions of inactive cyclin protease and kinase.

K_d, K_c, $K_j, j = 1, \ldots, 4$ - Michaelis constants

Sustained oscillations



Parameters

$$
\begin{aligned}
k_d &= 0.25 \\
v_i &= 0.25 \\
K_c &= 0.5 \\
V_{M1} &= 3 \\
V_2 &= 1.5 \\
K_d &= 0 \\
V_{M3} &= 1 \\
V_4 &= 0.7 \\
\varepsilon &= 10^{-3}
\end{aligned}
,
$$

$$
(1)\quad
\begin{aligned}
\frac{dC}{dt} &= \frac{1}{4}(1 - X - C) \\
\frac{dM}{dt} &= \frac{6C}{1+2C}\frac{1-M}{\varepsilon + 1 - M} - \frac{3}{2}\frac{M}{\varepsilon + M} \\
\frac{dX}{dt} &= M\frac{1-X}{\varepsilon + 1 - X} - \frac{7}{10}\frac{X}{\varepsilon + X}
\end{aligned}
$$

$$
\frac{X}{\varepsilon + X} = \begin{cases} \approx 1, & X = O(1) \\ \frac{\xi}{1+\xi}, & X = \varepsilon\xi \\ \approx 0, & X = o(\varepsilon) \end{cases}
$$

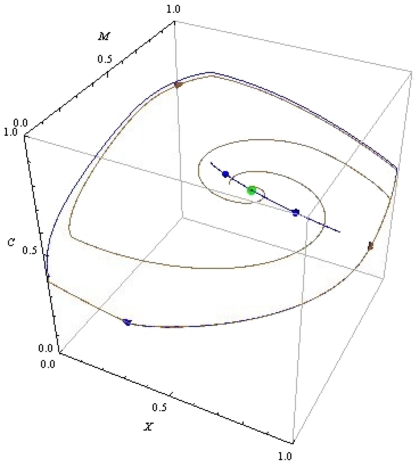
T. Erneux and A. Goldbeter, *Rescue of the quasi-steady-state approximation in a model for oscillations in an enzymatic cascade* SIAM J. Appl. Math 67 (2006), 305-320.

Limit cycle

Periodic orbit in the cube $[0,1]^3 \subset \mathbb{R}^3$

partially close to $X = 0$, $M = 1$, $X = 1$, $M = 0$

Theorem: For ε small there exists a strongly attracting periodic orbit Γ_ε of system (1) which tends to a singular cycle Γ_0 as $\varepsilon \to 0$.



Geometric Singular Perturbation Approach

(MO) as a singularly perturbed system of ODEs

$$X' = [M(1-X)(\varepsilon+X) - \tfrac{7}{10}X(\varepsilon+1-X)]F_\varepsilon(M)$$
$$M' = [\tfrac{6C}{1+2C}(1-M)(\varepsilon+M) - \tfrac{3}{2}M(\varepsilon+1-M)]F_\varepsilon(X)$$
$$C' = \tfrac{1}{4}(1-X-C)F_\varepsilon(M)F_\varepsilon(X)$$

obtained by multiplying (4) by the factor

$$F_\varepsilon(M)F_\varepsilon(X)) := (\varepsilon+1-M)(\varepsilon+M)(\varepsilon+1-X)(\varepsilon+X)$$

ε - singular perturbation parameter

non-standard form of slow-fast systems on fast time-scale

Layer problem

$$X' = \left(M - \frac{7}{10}\right) F_0(M, X)$$

$$M' = \left(\frac{6C}{1+2C} - \frac{3}{2}\right) F_0(M, X)$$

$$C' = 0.25(1 - X - C)F_0(M, X)$$

$$F_0(M, X) := (1 - M)M(1 - X)X$$

In the layer problem all three variables evolve!

Critical manifold S consists of four planes
$M = 0,\ M = 1,\ X = 0,\ X = 1$

$\exists$ single equilibrium point

Stability properties of the critical manifold S

Lemma. The layer problem has the following properties:

$X = 0$ is attracting for $M < 0.7$ and repelling for $M > 0.7$
$M = 1$ is attracting for $C > 0.5$ and repelling for $C < 0.5$
$X = 1$ is attracting for $M > 0.7$ and repelling for $M < 0.7$
$M = 0$ is attracting for $C < 0.5$ and repelling for $C > 0.5$
Equilibrium $(X, M, C) = (0.5, 0.7, 0.5)$ is of saddle-focus type

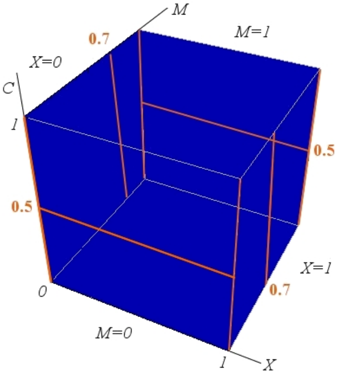
non-hyperbolic lines and edges

line $C = 0.5$
in the planes $M = 0$ and $M = 1$

line $M = 0.7$
in the planes $X = 0$ and $X = 1$

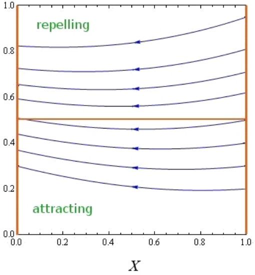
edges: $(C, 0, 0)$, $(C, 0, 1)$, $(C, 1, 0)$, and $(C, 1, 1)$ with $C \in [0, 1]$

Away from the nonhyperbolic lines and edges S perturbs to S_ε

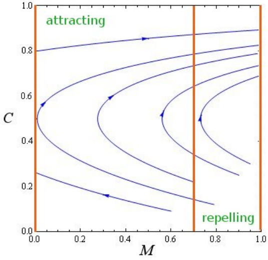


Slow dynamics

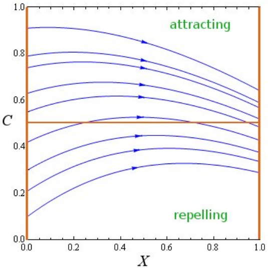
$M = 0$



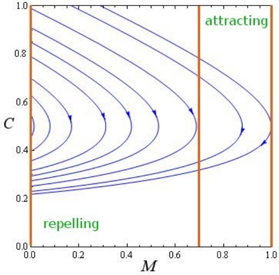
$X = 0$



$M = 1$



$X = 1$



relevant parts of the slow flow contract C

Periodic orbit

$\exists$ singular limit cycle Γ_0

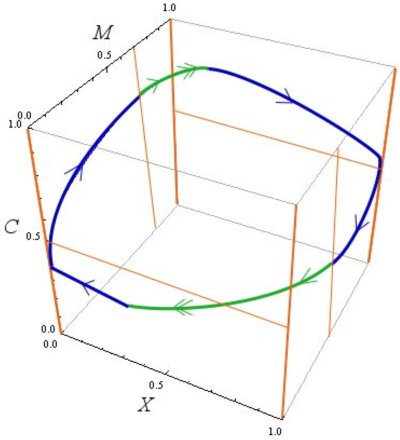
Slow motion:

in the attracting parts of the planes $M = 0$, $X = 0$, $M = 1$, $X = 1$

Exchange of stability at the edges

Fast jumps:

from $X = 0$ to $M = 1$
from $X = 1$ to $M = 0$

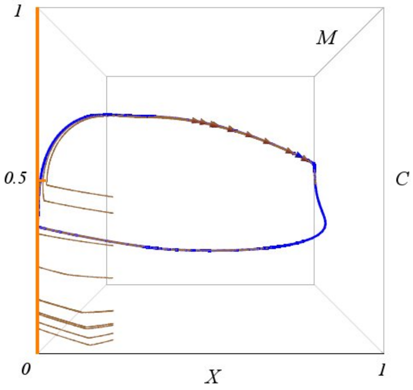


More details needed close to the edges!

Slow drift along the edge $(0, 0, C)$

Extended system

$$
\begin{aligned}
X' &= f_1(X, M, C, \varepsilon) \\
M' &= f_2(X, M, C, \varepsilon) \\
C' &= f_3(X, M, C, \varepsilon) \\
\varepsilon' &= 0
\end{aligned}
$$

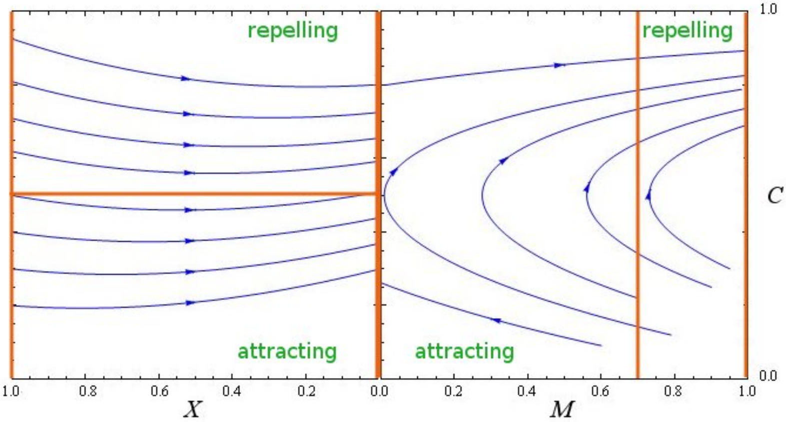


Edge $(0, 0, C, 0)$ - very degenerate!

Very slow drift along the edges $(X, M) = (0, 0)$ and $(X, M) = (1, 1)$ - studied the **blow-up** method!

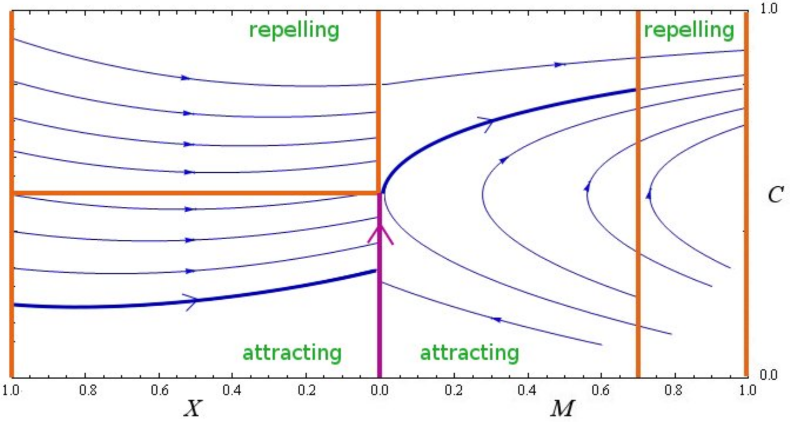
New phenomenon

$M = 0$ $\qquad\qquad$ $X = 0$



New phenomenon: "delayed" exchange of stability

$M = 0$ $X = 0$



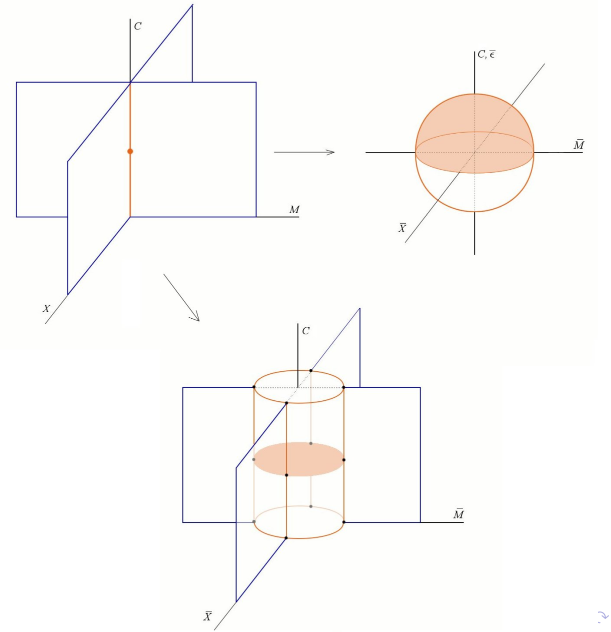
Very slow drift along the edge $(X, M) = (0, 0)$

Blow-up of the non-hyperbolic edge

$$
\begin{aligned}
X &= r\bar{X} \\
M &= r\bar{M}, \\
C &= \bar{C}, \\
\varepsilon &= r\bar{\varepsilon}
\end{aligned}
$$

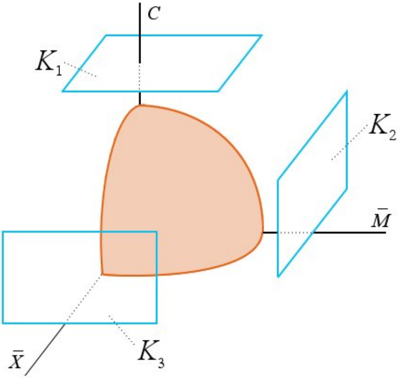
$$(\bar{X}, \bar{M}, \bar{\varepsilon}, \bar{C}) \in \mathbb{S}^2 \times \mathrm{I}$$

$$r \in \mathbb{R}$$



Charts

For C fixed each point $(0, 0, C)$ is blown-up to a sphere



$(\bar{X}, \bar{M}, \bar{\varepsilon}) \in \mathbb{S}^2$

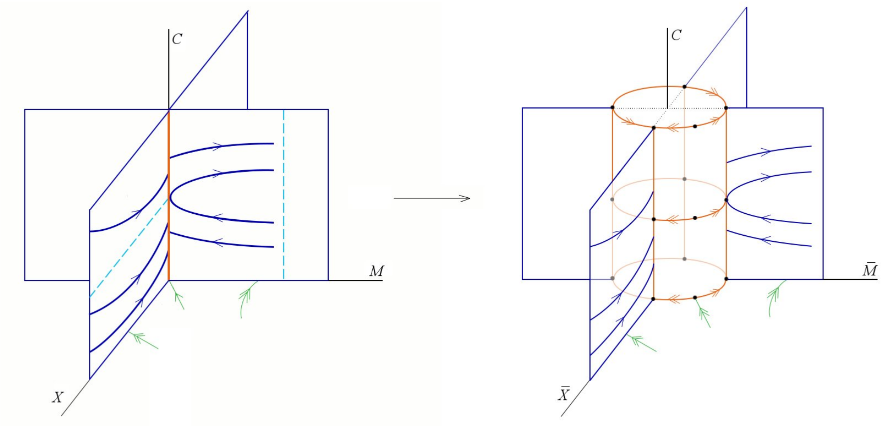
Charts

K_1: $\bar{\varepsilon} = 1$
(rescaling chart)

K_2: $\bar{M} = 1$

K_3: $\bar{X} = 1$

Blow-up of the non-hyperbolic edge



Dynamics in chart K_1

Slow-fast system with respect to ε

$$\begin{aligned} X' &= -0.7X(1+M) + O(\varepsilon) \\ M' &= [\tfrac{6C}{2C+1}(1+M) - \tfrac{3}{2}M](1+X) + O(\varepsilon) \\ C' &= 0.25(1-C)(1+M)(1+X)\varepsilon + O(\varepsilon^2) \end{aligned}$$

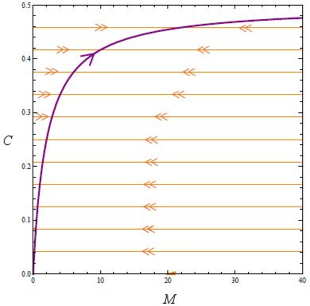
slow variable C, fast variables M, X

Critical manifold

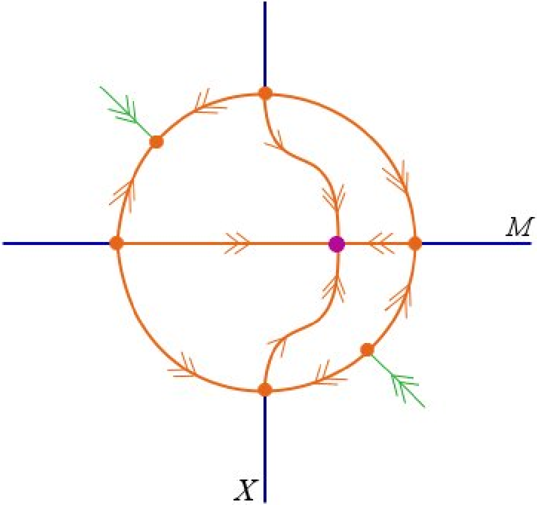
curve of equilibria

$X = 0, M = -\frac{4C}{2C-1}$

$C \in [0, 0.5)$, attracting



Dynamics of the blown-up system



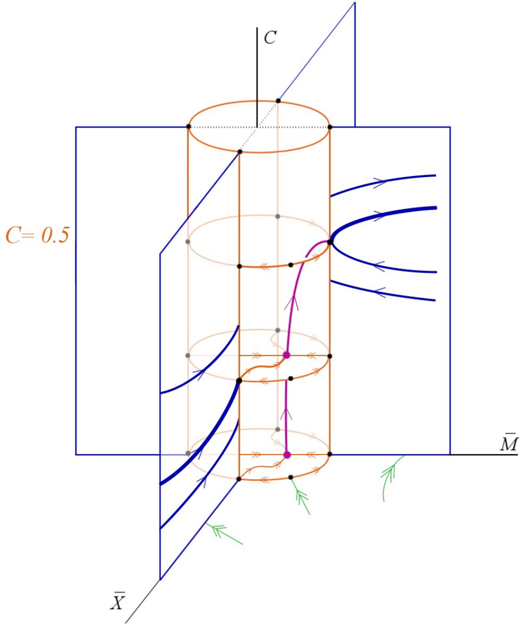
$\varepsilon = 0,\ 0 < C < 0.5$ - fixed

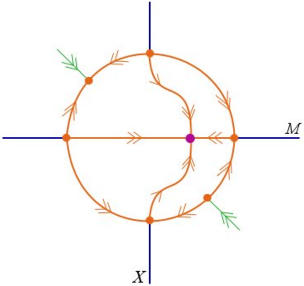
For fixed $C < 0.5$
equilibrium
$X = 0, M = -\frac{4C}{2C-1}$
is a stable node!

From the analysis in chart K_2**:**

for $C = 0.5$
two equilibria collide!

Dynamics of the blown-up system





Exit point at $C = 0.5$
still degenerate, second
blow-up needed!

Proof

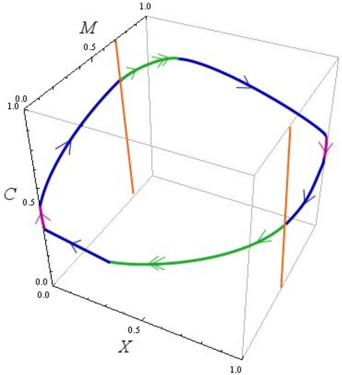
1. **singular cycle** Γ_0:

   - **slow motion** in $M = 0$

   - **very slow drift** along edge $(X, M) = (0, 0)$

   - **slow motion** in $X = 0$

   - **fast jump** at $M = 0.7$ from $X = 0$ to $M = 1$



2. Poincaré map close to the singular cycle: strongly contracting

Conclusion and Outlook

- case study 1: two-parameter singular perturbation problem, several scaling regimes
- case study 2: singular perturbation problem not in standard form
- singular behavior of critical manifold S is resolved by blow-up constructions
- use standard regular and singular perturbation results
- approach useful in other multi-parameter singular perturbation problems